Problem Sheet 1

1) Use Theorem 1.4 to prove that

$$\sum_{p\le x}\frac{1}{p}\ge\log\log x-1$$

for all **real** $x\ge 3$. This is a version of Theorem 1.4 with the *integer* N replaced by the *real* x.

Hint Given $x\ge 3$ let $N=[x]$, the largest integer $\le x$. Then, importantly, the sets of integers $\{n:1\le n\le x\}$ and $\{n:1\le n\le N\}$ are equal. Therefore sums (and products) over both sets are equal, i.e. for **any** terms a_n, $\sum_{n\le x}a_n=\sum_{n\le N}a_n$.

2) Corollary 1.2 states that

$$\log(N+1)\le\sum_{1\le n\le N}\frac{1}{n}\le\log N+1, \tag{18}$$

for *integer* N.

i. Prove that

$$\log N+\frac{1}{N}\le\sum_{1\le n\le N}\frac{1}{n}\le\log N+1$$

for $N\ge 1$.

ii. Why is the lower bound in Part i better than that in (18)?

iii. Prove that

$$\log x\le\sum_{1\le n\le x}\frac{1}{n}\le\log x+1$$

for all *real* $x\ge 1$.

The idea

$$(b-a)\,\underset{[a,b]}{\mathrm{glb}}f(t)\le\int_a^b f(t)\,dt\le(b-a)\,\underset{[a,b]}{\mathrm{lub}}f(t), \tag{19}$$

has lots of applications, and one of the most **important** is seen in the next question. When can a sum be replaced by an integral?

3) *Bounding a Sum by an Integral.*

Let f be a function integrable on $[M,N]$. Prove that for **integers** $N > M \geq 1$,

i) if f is *increasing*

$$\int_M^N f(t)\,dt + f(M) \leq \sum_{M\leq n\leq N} f(n) \leq \int_M^N f(t)\,dt + f(N)\,,$$

ii) if f is *decreasing*

$$\int_M^N f(t)\,dt + f(N) \leq \sum_{M\leq n\leq N} f(n) \leq \int_M^N f(t)\,dt + f(M)\,.$$

These two parts can be summed up by saying that if f is *monotonic* then

$$\min\left(f(M)\,,f(N)\right) \leq \sum_{M\leq n\leq N} f(n) - \int_M^N f(t)\,dt \leq \max\left(f(M)\,,f(N)\right),$$

for all integers $N > M \geq 1$.

Hint Apply (19).

4) i. Use Question 3 to prove that for integers $N \geq 1$

$$N\log N - (N-1) \leq \sum_{1\leq n\leq N} \log n \leq (N+1)\log N - (N-1)$$

ii Deduce that

$$e\left(\frac{N}{e}\right)^N \leq N! \leq eN\left(\frac{N}{e}\right)^N, \tag{20}$$

for $N \geq 1$.

This is a weak result, there is a factor of N difference between the upper and lower bounds. Which bound is closer to $N!$? Perhaps it lies somewhere in the middle? See later questions for the answers.

5) i) Use Question 3 to prove that for $\sigma > 0$, $\sigma \neq 1$

$$\frac{1}{N^\sigma} \leq \sum_{M\leq n\leq N} \frac{1}{n^\sigma} - \frac{N^{1-\sigma} - M^{1-\sigma}}{1-\sigma} \leq \frac{1}{M^\sigma}, \tag{21}$$

for all integers $N > M \geq 1$.

ii) You cannot substitute $\sigma = 1$ into part i because of the $1 - \sigma$ in the denominator but what is the limit

$$\lim_{\sigma \to 1} \frac{N^{1-\sigma} - M^{1-\sigma}}{1 - \sigma}?$$

What does the result (21) become under the limit $\sigma \to 1$?

6) Show that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{1}{\log x}, \tag{22}$$

for all real $x \geq 3$. Thus deduce that

$$\lim_{x \to \infty} \prod_{p \leq x} \left(1 - \frac{1}{p}\right) = 0.$$

In which case we say that the infinite product **diverges**.

Hint Replace x by an integer, look at the inverse of the product, and use ideas and results from the proof of Theorem 1.4.

Later in the course we will show that, with an appropriate constant c, we have

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{c}{\log x}$$

as $x \to \infty$, where $f(x) \sim g(x)$ as $x \to \infty$ means $\lim_{x \to \infty} f(x)/g(x) = 1$.

7) Let $\pi(x) = \sum_{p \leq x} 1$, the number of primes less than equal to x. The infinitude of primes is equivalent to $\lim_{x \to \infty} \pi(x) = \infty$

Prove that $\pi(x) \geq c \log x$ for some constant $c > 0$.

Justify each step in the following argument. With $D > 1$ a constant to be chosen

$$\begin{aligned}
\pi(x) &\geq \sum_{\frac{\log x}{D} < p \leq x} 1 \geq \sum_{\frac{\log x}{D} < p \leq x} \frac{\log x}{Dp} \\
&\geq \frac{\log x}{D} \left(\sum_{p \leq x} \frac{1}{p} - \sum_{2 \leq n \leq \frac{\log x}{D}} \frac{1}{n} \right) \\
&\geq \frac{\log x}{D} \left(\log\log x - 1 - \log\left(\frac{\log x}{D}\right)\right) \\
&\geq e^{-2} \log x,
\end{aligned}$$

for an appropriate choice of D. What is that choice and why?